Entanglement and Open Systems in Algebraic Quantum Field Theory

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Abstract

Entanglement has long been the subject of discussion by philosophers of quantum theory, and has recently come to play an essential role for physicists in their development of quantum information theory. In this paper we show how the formalism of algebraic quantum field theory (AQFT) provides a rigorous framework within which to analyze entanglement in the context of a fully relativistic formulation of quantum theory. What emerges from the analysis are new practical and theoretical limitations on an experimenter's ability to perform operations on a field in one spacetime region that can disentangle its state from the state of the field in other spacelike-separated regions. These limitations show just how deeply entrenched entanglement is in relativistic quantum field theory, and yield a fresh perspective on the ways in which the theory differs conceptually from both standard nonrelativistic quantum theory and classical relativistic field theory.

"...despite its conservative way of dealing with physical principles, algebraic QFT leads to a *radical change of paradigm*. Instead of the Newtonian view of a space-time filled with a material content one enters the reality of Leibniz created by relation (in particular inclusions) between 'monads' ($\sim$ the hyperfinite type III_1 local von Neumann factors $\mathcal{A}(O)$ which as single algebras are nearly void of physical meaning)" Schroer (1998, p. 302).

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1. Introduction

In *PCT, Spin and Statistics, and All That*, Streater and Wightman claim that, as a consequence of the axioms of algebraic quantum field theory (AQFT), "it is difficult to isolate a system described by fields from outside effects" (1989, p. 139). Haag makes a similar claim in *Local Quantum Physics*: "From the previous chapters of this book it is evidently not obvious how to achieve a division of the world into parts to which one can assign individuality... Instead we used a division according to regions in space-time. This leads in general to open systems" (1992, p. 298). By a field system these authors mean that portion of a quantum field within a specified bounded open region O of spacetime, with its associated algebra of observables $\mathcal{A}(O)$ (constructed in the usual way, out of field operators smeared with test functions having support in the region). The environment of a field system, so construed, is naturally taken to be the field in the region O', the spacelike complement of O. But then the claims above appear, at first sight, puzzling. After all, it is an axiom of AQFT that the observables in $\mathcal{A}(O')$ commute with those in $\mathcal{A}(O)$. And this implies — indeed, is *equivalent* to — the assertion that standard von Neumann measurements performed in O' *cannot* have 'outside effects' on the expectations of observables in O (Lüders, 1951). What, then, could the above authors possibly mean by saying that the field in O must be regarded as an open system?

A similar puzzle is raised by a famous passage in which Einstein (1948) contrasts the picture of physical reality embodied in classical field theories with that which emerges when we try to take quantum theory to be complete:

> "If one asks what is characteristic of the realm of physical ideas independently of the quantum theory, then above all the following attracts our attention: the concepts of physics refer to a real external world, i.e., ideas are posited of things that claim a "real existence" independent of the perceiving subject (bodies, fields, etc.)... it appears to be essential for this arrangement of the things in physics that, at a specific time, these things claim an existence independent of one another, insofar as these things "lie in different parts of space". Without such an assumption of the mutually independent existence (the "being-thus") of spatially distant things, an assumption which originates in everyday thought, physical thought in the sense familiar to us would not be possible. Nor does one see how physical laws could be formulated and tested without such clean separation.... For the relative independence of spatially distant things (A and B), this idea is characteristic: an external influence on A has no *immediate* effect on B; this is known as the "principle of local action," which is applied consistently in field theory. The complete suspension of this basic principle

> would make impossible the idea of the existence of (quasi-)closed systems and, thereby, the establishment of empirically testable laws in the sense familiar to us" (*ibid*, pp. 321-2; Howard's (1989) translation).

There is a strong temptation to read Einstein's 'assumption of the mutually independent existence of spatially distant things' and his 'principle of local action' as anticipating, respectively, the distinction between separability and locality, or between nonlocal 'outcome-outcome' correlation and 'measurement-outcome' correlation, which some philosophers argue is crucial to unravelling the conceptual implications of Bell's theorem (see, e.g., Howard 1989). However, even in nonrelativistic quantum theory, there is no question of any nonlocal *measurement*-outcome correlation between distinct systems or degrees of freedom, whose observables are always represented as commuting. Making the reasonable assumption that Einstein knew this quite well, what is it about taking quantum theory at face value that he saw as a threat to securing the existence of physically closed systems?

What makes quantum systems open for Einstein, as well as for Streater and Wightman, and Haag, is that they can occupy entangled states in which they sustain nonclassical EPR correlations with systems outside their light cones. That is, while it is correct to read Einstein's discussion of the mutually independent existence of distant systems as an implicit critique of the way in which quantum theory often represents their joint state as entangled, we believe it must be the *outcome-outcome* EPR correlations associated with entangled states that, in Einstein's view, pose a problem for the legitimate testing of the predictions of quantum theory. One could certainly doubt whether EPR correlations really pose any methodological problem, or whether they truly require the existence of physical (or 'causal') influences acting on a quantum system from outside. But the analogy with open systems in thermodynamics that Einstein and the others seem to be invoking is not entirely misplaced.

Consider the simplest toy universe consisting of two nonrelativistic quantum systems, represented by a tensor product of two-dimensional Hilbert spaces $\mathbb{C}^2_A \otimes \mathbb{C}^2_B$, where system A is the 'object' system, and B its 'environment'. Let x be any state vector for the composite system $A+B$, and $D_A(x)$ be the reduced density operator x determines for system A. Then the von Neumann entropy of A, $E_A(x) = -\mathrm{Tr}(D_A(x)\ln D_A(x))$ $(= E_B(x))$, varies with the degree to which A and B are entangled. If x is a product vector with no entanglement, $E_A(x) = 0$, whereas, at the opposite extreme, $E_A(x) = \ln 2$ when x is, say, a singlet or triplet state. The more A and B are entangled, the more 'disordered' A becomes, because it will have more than one state available to it and A's probabilities of occupying them will approach equality. In fact, exploiting an analogy to Carnot's heat cycle and the second law of thermodynamics — that it is impossible to construct a *perpetuum mobile* — Popescu and Rohrlich (1997) have shown that the general principle that it is impossible to cre-

ate entanglement between pairs of systems by local operations on one member of each pair implies that the von Neumann entropy of either member provides the uniquely correct measure of their entanglement when they are in a pure state. Changes in their degree of entanglement, and hence in the entropy of either system, can only come about in the presence of a nontrivial interaction Hamiltonian between them. But the fact remains that there is an intimate connection between a system's entanglement with its environment and the extent to which the system should be thought of as physically closed.

Returning to AQFT, Streater and Wightman, as well as Haag, all intend to make a far stronger claim about quantum field systems — a point that even applies to spacelike separated regions of a *free* field, and might well have offended Einstein's physical sensibilities even more. The point is that quantum field systems are *unavoidably* and *intrinsically* open to entanglement. Streater and Wightman's comment is made in reference to the Reeh-Schlieder (1961) theorem, a consequence of the general axioms of AQFT. We shall show that this theorem entails severe *practical* obstacles to isolating field systems from entanglement with other field systems. Haag's comment goes deeper, and is related to the fact that the algebras associated with field systems localized in spacetime regions are in all known models of the axioms type III von Neumann algebras. We shall show that this feature of the local algebras imposes a fundamental limitation on isolating field systems from entanglement even *in principle*.

Think again of our toy nonrelativistic universe $A+B$, with Alice in possession of system A, and the state x entangled. Although there are no operations that Alice can perform on system A which will reduce its entropy, she can still try to destroy its entanglement with B by performing a standard von Neumann measurement on A. If $P_{\pm}$ are the eigenprojections of the observable she measures, and the initial density operator of $A+B$ is $D = P_x$, where P_x is the projection onto the ray x generates, then the post-measurement joint state of $A+B$ will be given by the new density operator

$$D \to D' = (P_+ \otimes I)P_x(P_+ \otimes I) + (P_- \otimes I)P_x(P_- \otimes I). \tag{1}$$

Since the projections $P_{\pm}$ are one-dimensional, and x is entangled, there are nonzero vectors $a_x^{\pm} \in \mathbb{C}_A^2$ and $b_x^{\pm} \in \mathbb{C}_B^2$ such that $(P_{\pm} \otimes I)x = a_x^{\pm} \otimes b_x^{\pm}$, and a straightforward calculation reveals that D' may be re-expressed as

$$D' = \mathrm{Tr}[(P_+ \otimes I)P_x]P_+ \otimes P_{b_x^+} + \mathrm{Tr}[(P_- \otimes I)P_x]P_- \otimes P_{b_x^-}. \tag{2}$$

Thus, regardless of the initial state x, or the degree to which it was entangled, D' will always be a convex combination of product states, and there will no longer be any entanglement between A and B. One might say that Alice's operation on A has the effect of isolating A from any further EPR influences from B.

Moreover, this result can be generalized. Given any finite or infinite dimension for the Hilbert spaces H_A and H_B, there is always an operation Alice can perform

on A that will destroy its entanglement with B no matter what their initial state D was, pure or mixed. In fact, it suffices for Alice to measure any nondegenerate observable A with a discrete spectrum (excluding 0). The final state D' will then be a convex combination of product states, each of which is a product density operator obtained by 'collapsing' D using some particular eigenprojection of the measured observable. (The fact that disentanglement of a state can always be achieved in this way does not conflict with the recently established result there can be no 'universal disentangling machine', i.e., no *unitary* evolution that maps an arbitrary $A+B$ state D to an unentangled state with the same reduced density operators as D (Mor 1998; Mor and Terno 1999). Also bear in mind that we have *not* required that a successful disentangling process leave the states of the entangled subsystems unchanged.)

The upshot is that if entanglement *does* pose a methodological threat, it can at least be brought under control in nonrelativistic quantum theory. Not so when we consider the analogous setup in quantum field theory, with Alice in the vicinity of one region A, and B any other spacelike-seperated field system. We shall see that AQFT puts both practical and theoretical limits on Alice's ability to destroy entanglement between her field system and B. Again, while one could doubt whether this poses any real methodological problem for Alice (an issue to which we shall return later), we think it is ironic, considering Einstein's point of view, that such limits should be forced upon us precisely when we make the transition to a fully *relativistic* formulation of quantum theory.

We begin in Section 2. by reviewing the formalism of AQFT, the concept of entanglement between spacelike separated field systems, and the mathematical representation of an operation performed within a local spacetime region on a field system. In Section 3., we connect the Reeh-Schlieder theorem with the practical difficulties involved in guaranteeing that a field system is disentangled from other field systems. The language of operations also turns out to be indispensible for clearing up some apparently paradoxical physical implications of the Reeh-Schlieder theorem that have been raised in the literature without being properly resolved. In Section 4., we discuss differences between type III von Neumann algebras and the standard type I von Neumann algebras employed in nonrelativistic quantum theory, emphasizing the radical implications type III algebras have for the ignorance interpretation of mixtures and entanglement. We end Section 4. by connecting the type III character of the algebra of a local field system with the inability, in principle, to perform local operations on the system that will destroy its entanglement with other spacelike separated systems. We offer this result as one way to make precise the sense in which AQFT requires a radical change in paradigm — a change that, regrettably, has passed virtually unnoticed by philosophers of quantum theory.

2. AQFT, Entanglement, and Local Operations

First, let us recall that an abstract C^*-algebra is a Banach *-algebra, where the involution and norm are related by $|A^*A| = |A|^2$. Thus the algebra $\mathcal{B}(\mathsf{H})$ of all bounded operators on a Hilbert space H is a C^*-algebra, with * taken to be the adjoint operation, and $|\cdot|$ the standard operator norm. Moreover, any *-*sub*algebra of $\mathcal{B}(\mathsf{H})$ that is closed in the operator norm is a C^*-algebra, and, conversely, one can show that every abstract C^*-algebra has a concrete (faithful) representation as a norm-closed *-subalgebra of $\mathcal{B}(\mathsf{H})$, for some appropriate Hilbert space H (Kadison and Ringrose (henceforth, KR) 1997, Remark 4.5.7). On the other hand, a von Neumann algebra is always taken to be a concrete collection of operators on some fixed Hilbert space H. For F any set of operators on H, let F' denote the commutant of F, the set of all operators on H that commute with *every* operator in F. Observe that $F \subseteq F''$, that $F \subseteq G$ implies $G' \subseteq F'$, and (hence) that $A' = A'''$. $\mathcal{R}$ is called a von Neumann algebra exactly when $\mathcal{R}$ is a *-subalgebra of $\mathcal{B}(\mathsf{H})$ that contains the identity and satisfies $\mathcal{R} = \mathcal{R}''$. This is equivalent, via von Neumann's famous double commutant theorem (KR 1997, Theorem 5.3.1), to the assertion that $\mathcal{R}$ is closed in the strong operator topology, where $Z_n \to Z$ strongly just in case $|(Z_n - Z)x| \to 0$ for all $x \in \mathsf{H}$. If the sequence $\{Z_n\} \subseteq \mathcal{R}$ converges to $Z \in \mathcal{R}$ in norm, then since $|(Z_n - Z)x| \leq |Z_n - Z||x|$, the convergence is also strong, hence every von Neumann algebra is also a C^*-algebra. However, not every C^*-algebra of operators is a von Neumann algebra. For example, the C^*-algebra $\mathcal{C}$ of all compact operators on an infinite-dimensional Hilbert space H — that is, the norm closure of the *-subalgebra of all finite rank operators on H — does *not* contain the identity, nor does $\mathcal{C}$ satisfy $\mathcal{C} = \mathcal{C}''$. (Indeed, $\mathcal{C}'' = \mathcal{B}(\mathsf{H})$, because only multiples of the identity commute with all finite-dimensional projections, and of course *every* operator commutes with all multiples of the identity.) Finally, let S be any self-adjoint (i.e., *-closed) set of operators in $\mathcal{B}(\mathsf{H})$. Then S' is a *-algebra containing the identity, and both S' $(= S''' = (S')'')$ and S'' $(= (S')' = (S')''' = (S'')'')$ are von Neumann algebras. If we suppose there is some other von Neumann algebra $\mathcal{R}$ such that $S \subseteq \mathcal{R}$, then $\mathcal{R}' \subseteq S'$, which in turn entails $S'' \subseteq \mathcal{R}'' = \mathcal{R}$. Thus S'' is actually the smallest von Neumann algebra containing S, i.e., the von Neumann algebra that S generates. For example, the von Neumann algebra generated by all finite rank operators is the whole of $\mathcal{B}(\mathsf{H})$.

The basic mathematical object of AQFT on Minkowski spacetime M is an association $O \longmapsto \mathcal{A}(O)$ between bounded open subsets O of M and C^*-subalgebras $\mathcal{A}(O)$ of an abstract C^*-algebra $\mathcal{A}$ (Horuzhy 1988, Haag 1992). The motivation for this association is that the self-adjoint elements of $\mathcal{A}(O)$ represent the physical magnitudes, or observables, of the field intrinsic to the region O. We shall see below how the elements of $\mathcal{A}(O)$ can also be used to represent mathematically the physical operations that can be performed within O, and often it is only this latter interpretation of $\mathcal{A}(O)$

that is emphasized (Haag 1992, p. 104). One naturally assumes

Isotony: If $O_1 \subseteq O_2$, then $\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$.

As a consequence, the collection of all local algebras $\mathcal{A}(O)$ defines a net whose limit points can be used to define algebras associated with unbounded regions, and in particular $\mathcal{A}(M)$, which is identified with $\mathcal{A}$ itself.

One of the leading ideas in the algebraic approach to fields is that all of the physics of a particular field theory is encoded in the structure of its net of local algebras. (In particular, while any given field algebra on M obtained via smearing will define a unique net, the net underdetermines the field algebra; see Borchers 1960.) But there are some general assumptions about the net $\{\mathcal{A}(O) : O \subseteq M\}$ that all physically reasonable field theories are held to satisfy. First, one assumes

Microcausality: $\mathcal{A}(O') \subseteq \mathcal{A}(O)'$.

One also assumes that there is a faithful representation $\mathbf{x} \to \alpha_{\mathbf{x}}$ of the spacetime translation group of M in the group of automorphisms of $\mathcal{A}$, satisfying

Translation Covariance: $\alpha_{\mathbf{x}}(\mathcal{A}(O)) = \mathcal{A}(O + \mathbf{x})$.

Weak Additivity: For any $O \subseteq M$, $\mathcal{A}$ is the smallest C^*-algebra containing $\bigcup_{\mathbf{x}\in M} \mathcal{A}(O + \mathbf{x})$.

Finally, one assumes that there is some irreducible representation of the net $\{\mathcal{A}(O) : O \subseteq M\}$ in which these local algebras are identified with von Neumann algebras acting on a (nontrivial) Hilbert space H, $\mathcal{A}$ is identified with a strongly dense subset of $\mathcal{B}(\mathsf{H})$, and the following condition holds

Spectrum Condition: The generator of spacetime translations, the energy-momentum of the field, has a spectrum confined to the forward light-cone.

These last three conditions, and their role in the proof of the Reeh-Schlieder theorem (microcausality is not needed), are discussed at length in Halvorson (2000). We wish only to note here that while the spectrum condition itself only makes sense relative to a representation — wherein one can speak, via Stone's theorem, of a generator of the spacetime translation group of M (now concretely represented as a strongly continuous group of unitary operators $\{U_{\mathbf{x}}\}$ acting on H) — the requirement that the abstract net *have* a representation satisfying the spectrum condition does not require that one actually *pass* to such a representation to compute expectation values, cross-sections, etc. Indeed, Haag and Kastler (1964) have argued that there is a precise sense in which all concrete representations of a net are physically equivalent, including representations with and without a translationally invariant vacuum state

vector Ω. Since we are not concerned with that argument here, we shall henceforth take the 'Haag-Araki' approach of assuming that all the local algebras $\{\mathcal{A}(O) : O \subseteq M\}$ are von Neumann algebras acting on some H, with $\mathcal{A}'' = \mathcal{B}(\mathsf{H})$, and there is a translationally invariant vacuum state $\Omega \in \mathsf{H}$.

We turn next to the concept of a state of the field. Generally, a physical state of a quantum system represented by some von Neumann algebra $\mathcal{R} \subseteq \mathcal{B}(\mathsf{H})$ is given by a normalized linear expectation functional τ on $\mathcal{R}$ that is both positive and countably additive. Positivity is the requirement that τ map any positive operator in $\mathcal{R}$ to a nonnegative expectation (a must, given that positive operators have nonnegative spectra), while countable additivity is the requirement that τ be additive over countable sums of mutually orthogonal projections in $\mathcal{R}$. (There are also non-countably additive or 'singular' states on $\mathcal{R}$ (KR 1997, p. 723), but whenever we use the term 'state' we shall mean *countably additive* state.) Every state on $\mathcal{R}$ extends to a state ρ on $\mathcal{B}(\mathsf{H})$ which, in turn, can be represented by a density operator D_ρ on H via the standard formula $\rho(\cdot) = \mathrm{Tr}(D_\rho \cdot)$ (KR 1997, p. 462). A pure state on $\mathcal{B}(\mathsf{H})$, i.e., one that is not a nontrivial convex combination or mixture of other states of $\mathcal{B}(\mathsf{H})$, is then represented by a vector $x \in \mathsf{H}$. We shall always use the notation ρ_x for the normalized state functional $(x, \cdot x)/|x|^2$ $(= \mathrm{Tr}(P_x \cdot))$. If, furthermore, we consider the restriction $\rho_x|_{\mathcal{R}}$, the induced state on some von Neumann subalgebra $\mathcal{R} \subseteq \mathcal{B}(\mathsf{H})$, we cannot in general expect it to be pure on $\mathcal{R}$ as well. For example, with $\mathsf{H} = \mathbb{C}^2_A \otimes \mathbb{C}^2_B$, $\mathcal{R} = \mathcal{B}(\mathbb{C}^2_A) \otimes I$, and x entangled, we know that the induced state $\rho_x|_{\mathcal{R}}$, represented by $D_A(x) \in \mathcal{B}(\mathbb{C}^2_A)$, is *always* mixed. Similarly, one cannot expect that a pure state ρ_x of the field algebra $\mathcal{A}'' = \mathcal{B}(\mathsf{H})$ — which supplies a maximal specification of the state of the field *throughout* spacetime — will have a restriction to a local algebra $\rho_x|_{\mathcal{A}(O)}$ that is itself pure. In fact, we shall see later that the Reeh-Schlieder theorem entails that the vacuum state's restriction to any local algebra is always highly mixed.

There are two topologies on the state space of a von Neumann algebra $\mathcal{R}$ that we shall need to invoke. One is the metric topology induced by the norm on linear functionals. The norm of a state ρ on $\mathcal{R}$ is defined by $\|\rho\| \equiv \sup\{|\rho(Z)| : Z = Z^* \in \mathcal{R},\ |Z| \leq 1\}$. If two states, ρ_1 and ρ_2, are close to each other in norm, then they dictate close expectation values uniformly for *all* observables. In particular, if both ρ_1 and ρ_2 are vector states, i.e., they are induced by vectors $x_1, x_2 \in \mathsf{H}$ such that $\rho_1 = \rho_{x_1}|_{\mathcal{R}}$ and $\rho_2 = \rho_{x_2}|_{\mathcal{R}}$, then $|x_1 - x_2| \to 0$ implies $\|\rho_1 - \rho_2\| \to 0$. (It is important not to conflate the terms 'vector state' and 'pure state', unless of course $\mathcal{R} = \mathcal{B}(\mathsf{H})$ itself.) More generally, whenever the trace norm distance between two density operators goes to zero, the norm distance between the states they induce on $\mathcal{R}$ goes to zero. Since every state on $\mathcal{B}(\mathsf{H})$ is given by a density operator, which in turn can be decomposed as an infinite convex combination of one dimensional projections (with the infinite sum understood as trace norm convergence), it follows that every state on $\mathcal{R} \subseteq \mathcal{B}(\mathsf{H})$ is the norm limit of convex combinations of vectors states of

$\mathcal{R}$ (cf. KR 1997, Thm. 7.1.12). The other topology we shall invoke is the weak-* topology: a sequence or net of states $\{\rho_n\}$ on $\mathcal{R}$ weak-* converges to a state ρ just in case $\rho_n(Z) \to \rho(Z)$ for all $Z \in \mathcal{R}$. This convergence need not be uniform on all elements of $\mathcal{R}$, and is therefore weaker than the notion of approximation embodied by norm convergence. As it happens, any state on $\mathcal{B}(\mathsf{H})$ that is the weak-* limit of a set of states is also their norm limit, but this is only true for type I von Neumann algebras (Connes and Størmer, 1978, Cor. 9).

Next, we turn to defining entanglement in a field. Fix a state ρ on $\mathcal{B}(\mathsf{H})$, and two mutually commuting subalgebras $\mathcal{R}_A, \mathcal{R}_B \subseteq \mathcal{B}(\mathsf{H})$. To define what it means for ρ to be entangled across the algebras, we need only consider the restriction $\rho|_{\mathcal{R}_{AB}}$ to the von Neumann algebra they generate, $\mathcal{R}_{AB} = [\mathcal{R}_A \cup \mathcal{R}_B]''$, and of course we need a definition that also applies when $\rho|_{\mathcal{R}_{AB}}$ is mixed. A state ω on $\mathcal{R}_{AB}$ is called a product state just in case there are states ω_A of $\mathcal{R}_A$ and ω_B of $\mathcal{R}_B$ such that $\omega(XY) = \omega_A(X)\omega_B(Y)$ for all $X \in \mathcal{R}_A$, $Y \in \mathcal{R}_B$. Clearly, product states, or convex combinations of product states, possess only classical correlations. Moreover, if one can even just *approximate* a state with convex combinations of product states, its correlations do not significantly depart from those characteristic of a classical statistical theory. Therefore, we define ρ to be entangled across $(\mathcal{R}_A, \mathcal{R}_B)$ just in case $\rho|_{\mathcal{R}_{AB}}$ is *not* a weak-* limit of convex combinations of product states of $\mathcal{R}_{AB}$ (Halvorson and Clifton, 2000). Notice that we chose weak-* convergence rather than convergence in norm, hence we obtain a strong notion of entanglement. In the case $\mathsf{H} = \mathsf{H}_A \otimes \mathsf{H}_B$, $\mathcal{R}_A = \mathcal{B}(\mathsf{H}_A) \otimes I$, and $\mathcal{R}_B = I \otimes \mathcal{B}(\mathsf{H}_B)$, the definition obviously coincides with the usual notion of entanglement for a pure state (convex combinations and approximations being irrelevant in that case), and also coincides with the definition of entanglement (usually called 'nonseparability') for a mixed density operator that is standard in quantum information theory (Werner, 1989; Clifton and Halvorson, 2000; Clifton *et al*, 2000). Further evidence that the definition captures an essentially nonclassical feature of correlations is given by the fact that $\mathcal{R}_{AB}$ will possess an entangled state in the sense defined above if and *only if* both $\mathcal{R}_A$ and $\mathcal{R}_B$ are nonabelian (Bacciagaluppi, 1993, Thm. 7; Summers and Werner, 1995, Lemma 2.1). Returning to AQFT, it is therefore reasonable to say that a global state of the field ρ on $\mathcal{A}'' = \mathcal{B}(\mathsf{H})$ is entangled across a pair of spacelike-separated regions (O_A, O_B) just in case $\rho|_{\mathcal{A}_{AB}}$, ρ's restriction to $\mathcal{A}_{AB} = [\mathcal{A}(O_A) \cup \mathcal{A}(O_B)]''$, falls outside the weak-* closure of the convex hull of $\mathcal{A}_{AB}$'s product states.

Our next task is to review the mathematical representation of operations, highlight some subtleties in their physical interpretation, and then discuss what is meant by *local* operations on a system. We then end this section by giving the general argument that local operations performed in either of two spacelike separated regions (O_A,O_B) cannot create entanglement in a state across the regions.

The most general transformation of the state of a quantum system with Hilbert

space H is described by an operation on $\mathcal{B}(\mathsf{H})$, defined to be a positive, weak-* continuous, linear map $T : \mathcal{B}(\mathsf{H}) \to \mathcal{B}(\mathsf{H})$ satisfying $0 \leq T(I) \leq I$ (Haag and Kastler, 1964; Davies, 1976; Kraus, 1983; Busch *et al*, 1995; Werner 1987). (The weak-* topology on a von Neumann algebra $\mathcal{R}$ is defined in complete analogy to the weak-* topology on its state space, i.e., $\{Z_n\} \subseteq \mathcal{R}$ weak-* converges to $Z \in \mathcal{R}$ just in case $\rho(Z_n) \to \rho(Z)$ for all states ρ of $\mathcal{R}$.) Any such T induces a map $\rho \to \rho^T$ from the state space of $\mathcal{B}(\mathsf{H})$ into itself or 0, where, for all $Z \in \mathcal{B}(\mathsf{H})$,

$$\rho^T(Z) \equiv \rho(T(Z))/\rho(T(I)) \ \text{ if } \rho(T(I)) \neq 0; \ \ \equiv 0 \text{ otherwise.} \tag{3}$$

The number $\rho(T(I))$ is the probability that an ensemble in state ρ will respond 'Yes' to the question represented by the positive operator $T(I)$. An operation T is called selective if $T(I) < I$, and nonselective if $T(I) = I$. The final state after a selective operation on an ensemble of identically prepared systems is obtained by ignoring those members of the ensemble that fail to respond 'Yes' to $T(I)$. Thus a selective operation involves performing a physical operation on an ensemble followed by a *purely conceptual* operation in which one makes a selection of a subensemble based on the outcome of the physical operation (assigning 'state' 0 to the remainder). Nonselective operations, by contrast, always elicit a 'Yes' response from any state, hence the final state is not obtained by selection but purely as a result of the physical interaction between object system and the device that effects the operation. (We shall shortly discuss some actual physical examples to make this general description of operations concrete.)

An operation T, which quantum information theorists call a superoperator (acting, as it does, on operators to produce operators), "can describe any combination of unitary operations, interactions with an ancillary quantum system or with the environment, quantum measurement, classical communication, and subsequent quantum operations conditioned on measurement results" (Bennett *et al*, 1999). Interestingly, a superoperator itself can always be represented in terms of operators, as a consequence of the Kraus representation theorem (1983, p. 42): for any operation $T : \mathcal{B}(\mathsf{H}) \to \mathcal{B}(\mathsf{H})$, there exists a (not necessarily unique) countable collection of Kraus operators $\{K_i\} \subseteq \mathcal{B}(\mathsf{H})$ such that

$$T(\cdot) = \sum_i K_i^*(\cdot)K_i, \text{ with } 0 \leq \sum_i K_i^* K_i \leq I \tag{4}$$

where both sums, if infinite, are to be understood as weak-* convergence. It is not difficult to show that the sum $\sum_i K_i K_i^*$ must also weak-* converge, hence we can let T^* denote the operation conjugate to T whose Kraus operators are $\{K_i^*\}$. It then follows (using the linearity and cyclicity of the trace) that if a state ρ is represented by a density operator D on H, ρ^T will be represented by the density operator $T^*(D)$.

If the mapping $\rho \to \rho^T$, or equivalently, $D \to T^*(D)$, maps pure states to pure states, then the operation T is called a pure operation, and this corresponds to it being representable by a *single* Kraus operator.

More generally, the Kraus representation shows that a general operation is always equivalent to mixing the results of separating an initial ensemble into subensembles to which one applies pure (possibly selective) operations, represented by the individual Kraus operators. To see this, let T be an arbitrary operation performed on a state ρ, where $\rho^T \neq 0$, and suppose T is represented by Kraus operators $\{K_i\}$. Let ρ^{K_i} denote the result of applying to ρ the pure operation given by the mapping $T_i(\cdot) = K_i^*(\cdot)K_i$, and (for convenience) define $\lambda_i = \rho(T_i(I))/\rho(T(I))$. Then, at least when there are finitely many Kraus operators, it is easy to see that T itself maps ρ to the convex combination $\rho^T = \sum_i \lambda_i \rho^{K_i}$. In the infinite case, this sum converges not just weak-* but *in norm*, and it is a useful exercise in the topologies we have introduced to see why. Letting ρ_n^T denote the partial sum $\sum_{i=1}^n \lambda_i \rho^{K_i}$, we need to establish that

$$\lim_{n\to\infty}[\sup\{|\rho^T(Z) - \rho_n^T(Z)| : Z = Z^* \in \mathcal{B}(\mathsf{H}),\ |Z| \leq 1\}] = 0. \tag{5}$$

For *any* $Z \in \mathcal{B}(\mathsf{H})$, we have

$$|\rho^T(Z) - \rho_n^T(Z)| = \rho(T(I))^{-1}|\sum_{i=n+1}^{\infty} \rho(K_i^* Z K_i)|. \tag{6}$$

However, $\rho(K_i^*(\cdot)K_i)$, being a positive linear functional, has a norm that may be computed by its action on the identity (KR 1997, Thm. 4.3.2). Therefore, $|\rho(K_i^* Z K_i)| \leq |Z|\rho(K_i^* K_i)$, and we obtain

$$|\rho^T(Z) - \rho_n^T(Z)| \leq \rho(T(I))^{-1}|Z| \sum_{i=n+1}^{\infty} \rho(K_i^* K_i). \tag{7}$$

However, since $\sum_i K_i^* K_i$ weak-* converges, this last summation is just the tail set of a convergent series. Therefore, when $|Z| \leq 1$, the right-hand side of (7) goes to zero independently of Z.

To get a concrete idea of how operations work physically, and to highlight two important interpretational pitfalls, let us again consider our toy universe, with $\mathsf{H} = \mathbb{C}_A^2 \otimes \mathbb{C}_B^2$ and x an entangled state. Recall that Alice disentangled x by measuring an A observable with eigenprojections $P_\pm$. Her measurement corresponds to applying the nonselective operation T with Kraus operators $K_1 = P_+ \otimes I$ and $K_2 = P_- \otimes I$, resulting in the final state $T^*(P_x) = T(P_x) = D'$, as given in (1). If Alice were to further 'apply' the pure selective operation T' represented by the single Kraus operator $P_+ \otimes I$, the final state of her ensemble, as is apparent from (2), would be

the product state $D'' = P_+ \otimes P_{b_x^+}$. But, as we have emphasized, this corresponds to a conceptual operation in which Alice just throws away all members of the original ensemble that yielded measurement outcome -1. On the other hand, it is essential not lose sight of the issue that troubled Einstein. *Whatever* outcome Alice selects for, she will then be in a position to assert that certain B observables — those that have either b_x^+ or b_x^- as an eigenvector, depending on the outcome she favours — have a sharp value in the ensemble she is left with. But prior to Alice performing the first operation T, such an assertion would have contradicted the orthodox interpretation of the entangled superposition x. If, contra Bohr, one were to view this change in B's state as a *real physical* change brought about by one of the operations Alice performs, surely the innocuous conceptual operation T' could not be the culprit — it must have been T which forced B to 'choose' between the alternatives $b_x^{\pm}$. Unfortunately, this clear distinction between the physical operation T and conceptual operation T' is not reflected well in the formalism of operations. For we could equally well have represented Alice's final product state $D'' = P_+ \otimes P_{b_x^+}$, not as the result of successively applying the operations T and T', but as the outcome of applying the single composite operation $T' \circ T$, which is just the mapping T'. And *this* T' now needs to be understood, not purely as a conceptual operation, but as also involving a physical operation, with possibly real nonlocal effects on B, depending on one's view of the EPR paradox. (In particular, keep in mind that you are taking the first step on the road to conceding the incompleteness of quantum theory if you attribute the change in the state of B brought about by T' in this case to a mere change in Alice's *knowledge* about B's state.)

There is a second pitfall that concerns interpreting the result of *mixing* subensembles, as opposed to singling out a particular subensemble. Consider an alternative method available to Alice for disentangling a state x. For concreteness, let us suppose that x is the singlet state $1/\sqrt{2}(a^+ \otimes b^- - a^- \otimes b^+)$. Alice applies the nonselective operation with Kraus representation

$$T(\cdot) = \frac{1}{2}(\sigma_a \otimes I)(\cdot)(\sigma_a \otimes I) + \frac{1}{2}(I \otimes I)(\cdot)(I \otimes I), \tag{8}$$

where σ_a is the spin observable with eigenstates $a^{\pm}$. Since $\sigma_a \otimes I$ maps x to the triplet state $1/\sqrt{2}(a^+ \otimes b^- + a^- \otimes b^+)$, T^* $(= T)$ will map P_x to an equal mixture of the singlet and triplet, which admits the following convex decomposition into product states

$$D' = \frac{1}{2}P_{a^+ \otimes b^-} + \frac{1}{2}P_{a^- \otimes b^+}. \tag{9}$$

Has Alice truly disentangled A from B? Technically, Yes. Yet all Alice has done, physically, is to separate the initial A ensemble into two subensembles in equal proportion, left the second subensemble alone while performing a (pure, nonselective)

unitary operation $\sigma_a \otimes I$ on the first that maps all its $A+B$ pairs to the triplet state, and then remixed the ensembles. Thus, notwithstanding the above decomposition of the final density matrix D', Alice *knows quite well* that she is in possession of an ensemble of A systems each of which is entangled either via the singlet or triplet state with the corresponding B systems. This will of course be recognized as one aspect of the problem with the ignorance interpretation of mixtures. We have two different ways to decompose D' — as an equal mixture of the singlet and triplet or of two product states — but which is the correct way to understand how the ensemble is *actually* constituted? The definition of entanglement is just not sensitive to the answer. (It is exactly this insensitivity that is at the heart of the recent dispute over whether NMR quantum computing is correctly understood as implementing genuine *quantum* computing that cannot be simulated classically (Braunstein *et al*, 1999; Laflamme, 1998).) Nevertheless, we are inclined to think the destruction of the singlet's entanglement that Alice achieves by applying the operation in (8) is an artifact of her mixing process, in which she is represented as simply forgetting about the history of the A systems. And this is the view we shall take when we consider similar possibilities for destroying entanglement between field systems in AQFT.

In the two examples considered above, Alice applies operations whose Kraus operators lie in the subalgebra $\mathcal{B}(\mathsf{H}_A) \otimes I$ associated with system A. In the case of a nonselective operation, this is clearly sufficient for her operation not to have any effect on the expectations of the observables of system B. However, it is also necessary. The point is quite general. Let us define a nonselective operation T to be (*pace* Einstein!) local to the subsystem represented by a von Neumann subalgebra $\mathcal{R} \subseteq \mathcal{B}(\mathsf{H})$ just in case $\rho^T|_{\mathcal{R}'} = \rho|_{\mathcal{R}'}$ for all states ρ. Thus, we require that T leave the expectations of observables outside of $\mathcal{R}$, as well as those in its center $\mathcal{R} \cap \mathcal{R}'$, unchanged. Since distinct states of $\mathcal{R}'$ cannot agree on all expectation values, this means T must act like the identity operation on $\mathcal{R}'$. Now fix an arbitrary element $Y \in \mathcal{R}'$, and suppose T is represented by Kraus operators $\{K_i\}$. A straightforward calculation reveals that

$$\sum_i [Y, K_i]^*[Y, K_i] = T(Y^2) - T(Y)Y - YT(Y) + YT(I)Y. \tag{10}$$

Since $T(I) = I$, and T leaves the elements of $\mathcal{R}'$ fixed, the right-hand side of (10) reduces to zero. Thus each of the terms in the sum on the left-hand side, which are positive operators, must individually be zero. Since Y was an arbitrary element of $\mathcal{R}'$, it follows that $\{K_i\} \subseteq (\mathcal{R}')' = \mathcal{R}$. So we see that nonselective operations local to $\mathcal{R}$ *must* be represented by Kraus operators taken from the subalgebra $\mathcal{R}$.

As for selective operations, we have already seen that they *can* 'change' the global statistics of a state ρ outside the subalgebra $\mathcal{R}$, particularly when ρ is entangled. However, a natural extension of the definition of local operation on $\mathcal{R}$ to a cover the case when T is selective is to require that $T(Y) = T(I)Y$ for all $Y \in \mathcal{R}'$. This implies

$\rho^T(Y) = \rho(T(I)Y)/\rho(T(I))$, and so guarantees that T will leave the statistics of any observable in $\mathcal{R}'$ the same *modulo* whatever correlations that observable might have had in the initial state with the Yes/No question represented by the positive operator $T(I)$. Further motivation is provided by the fact this definition is equivalent to requiring that T factor across the algebras $(\mathcal{R}, \mathcal{R}')$, in the sense that $T(XY) = T(X)Y$ for all $X \in \mathcal{R}$, $Y \in \mathcal{R}'$ (Werner, 1987, Lemma). If there exist product states across $(\mathcal{R}, \mathcal{R}')$ (an assumption we shall later see does *not* usually hold when $\mathcal{R}$ is a local algebra in AQFT), this guarantees that any local selective operation on $\mathcal{R}$, when the global state is an entirely uncorrelated product state, will leave the statistics of that state on $\mathcal{R}'$ unchanged. Finally, observe that $T(Y) = T(I)Y$ for all $Y \in \mathcal{R}'$ implies that the right-hand side of (10) again reduces to zero. Thus it follows (as before) that selective local operations on $\mathcal{R}$ must also be represented by Kraus operators taken from the subalgebra $\mathcal{R}$.

Applying these considerations to field theory, any local operation on the field system within a region O, whether or not the operation is selective, is represented by a family of Kraus operators taken from $\mathcal{A}(O)$. In particular, each individual element of $\mathcal{A}(O)$ represents a pure operation that can be performed within O (cf. Haag and Kastler, 1964, p. 850). We now need to argue that local operations performed by two experimenters in spacelike separated regions cannot create entanglement in a state across the regions where it had none before. This point, well-known by quantum information theorists working in nonrelativistic quantum theory, in fact applies quite generally to any two commuting von Neumann algebras $\mathcal{R}_A$ and $\mathcal{R}_B$.

Suppose that a state ρ is not entangled across $(\mathcal{R}_A, \mathcal{R}_B)$, local operations T_A and T_B are applied to ρ, and the result is nonzero (i.e., some members of the initial ensemble are not discarded). Since the Kraus operators of these operations commute, it is easy to check that $(\rho^{T_A})^{T_B} = (\rho^{T_B})^{T_A}$, so it does not matter in which order we take the operations. It is sufficient to show that ρ^{T_A} will again be unentangled, for then we can just repeat the same argument to obtain that neither can $(\rho^{T_A})^{T_B}$ be entangled. Next, recall that a general operation T_A will just produce a mixture over the results of applying a countable collection of pure operations to ρ; more precisely, the result will be the norm, and hence weak-*, limit of finite convex combinations of the results of applying pure operations to ρ. If the states that result from ρ under those pure operations are themselves not entangled, ρ^{T_A} itself could not be either, because the set of unentangled states is by definition convex and weak-* closed. Without loss of generality, then, we may assume that the local operation T_A is pure and, hence, given by $T_A(\cdot) = K^*(\cdot)K$, for some *single* Kraus operator $K \in \mathcal{R}_A$. As before, we shall denote the resulting state ρ^{T_A} by ρ^K $(\equiv \rho(K^* \cdot K)/\rho(K^*K))$.

Next, suppose that ω is any product state on $\mathcal{R}_{AB}$ with restrictions to $\mathcal{R}_A$ and

$\mathcal{R}_B$ given by ω_A and ω_B, and such that $\omega^K \neq 0$. Then, for any $X \in \mathcal{R}_A$, $Y \in \mathcal{R}_B$,

$$\omega^K(XY) = \frac{\omega(K^*(XY)K)}{\omega(K^*K)} \tag{11}$$

$$= \frac{\omega(K^*XKY)}{\omega(K^*K)} \tag{12}$$

$$= \frac{\omega_A(K^*XK)}{\omega_A(K^*K)}\omega_B(Y) = \omega_A^K(X)\omega_B(Y). \tag{13}$$

It follows that K maps product states of $\mathcal{R}_{AB}$ to product states (or to zero). Suppose, instead, that ω is a convex combination of states on $\mathcal{R}_{AB}$, $\omega = \sum_{i=1}^n \lambda_i \omega_i$. Then, setting $\lambda_i^K = \omega_i(K^*K)/\omega(K^*K)$, it is easy to see that $\omega^K = \sum_{i=1}^n \lambda_i^K \omega_i^K$, hence K preserves convex combinations of states on $\mathcal{R}_{AB}$ as well. It is also not difficult to see that the mapping $\omega \mapsto \omega^K$ is weak-* continuous at any point where $\omega^K \neq 0$ (cf. Halvorson and Clifton, 2000, Sec. 3). Returning to our original state ρ, our hypothesis is that it is not entangled. Thus, there is a net of states $\{\omega_n\}$ on $\mathcal{R}_{AB}$, each of which is a convex combination of product states, such that $\omega_n \to \rho|_{\mathcal{R}_{AB}}$ in the weak-* topology. It follows from the above considerations that $\omega_n^K \to \rho^K|_{\mathcal{R}_{AB}}$, where each of the states $\{\omega_n\}$ is again a convex combination of product states. Hence $\rho^K|_{\mathcal{R}_{AB}}$ is not entangled either.

3. The Operational Implications of the Reeh-Schlieder Theorem

Again, let $\mathcal{R} \subseteq \mathcal{B}(\mathsf{H})$ be any von Neumann algebra. A vector $x \in \mathsf{H}$ is called cyclic for $\mathcal{R}$ if the norm closure of the set $\{Ax : A \in \mathcal{R}\}$ is the *whole* of H. In AQFT, the Reeh-Schlieder (RS) theorem connects this formal property of cyclicity to the physical property of a field state having bounded energy. (More generally, the connection is between cyclicity and field states that are 'analytic' in the energy. This, together with the physical and mathematical origins of the RS theorem, are analyzed in depth in Halvorson (2000).) A pure global state x of the field has bounded energy just in case $E([0,r])x = x$ for some $r < \infty$, where E is the spectral measure for the global Hamiltonian of the field. In other words, the probability in state x that the field's energy is confined to the bounded interval $[0,r]$ is unity. In particular, the vacuum Ω is an eigenstate of the Hamiltonian with eigenvalue 0, and hence trivially has bounded energy. The RS theorem implies that

If x has bounded energy, then x is cyclic for any local algebra $\mathcal{A}(O)$.

Our first order of business is to explain Streater and Wightman's comment that the RS theorem entails "it is difficult to isolate a system described by fields from outside effects" (1989, p. 139).

A vector x is called separating for a von Neumann algebra $\mathcal{R}$ if $Ax = 0$ implies $A = 0$ whenever $A \in \mathcal{R}$. It is an elementary result of von Neumann algebra theory that x will be cyclic for $\mathcal{R}$ if and only if x is separating for $\mathcal{R}'$ (KR 1997, Cor. 5.5.12). To illustrate with a simple example, take $\mathsf{H} = \mathsf{H}_A \otimes \mathsf{H}_B$. If $\dim \mathsf{H}_A \geq \dim \mathsf{H}_B$, then it is possible for there to be vectors $x \in \mathsf{H}$ that have a Schmidt decomposition $\sum_i c_i a_i \otimes b_i$ where $|c_i|^2 \neq 0$ for *all* $i = 1$ to $\dim \mathsf{H}_B$. If we act on such an x by an operator in the subalgebra $I \otimes \mathcal{B}(\mathsf{H}_B)$, of form $I \otimes B$, then the only way $(I \otimes B)x$ can be the zero vector is if B itself maps all the basis vectors $\{b_i\}$ to zero, i.e., $I \otimes B = 0$. Thus such vectors are separating for $I \otimes \mathcal{B}(\mathsf{H}_B)$, and therefore cyclic for $\mathcal{B}(\mathsf{H}_A) \otimes I$. Conversely, it is easy to convince oneself that $\mathcal{B}(\mathsf{H}_A) \otimes I$ possesses a cyclic vector — equivalently, $I \otimes \mathcal{B}(\mathsf{H}_B)$ has a separating vector — *only if* $\dim \mathsf{H}_A \geq \dim \mathsf{H}_B$. So, to take another example, each of the A and B subalgebras will possess a cyclic and a separating vector just in case H_A and H_B have the same dimension (cf. the proof of Clifton *et al* 1998, Thm. 4).

Consider, now, a local algebra $\mathcal{A}(O)$ with $O' \neq \emptyset$, and a field state x with bounded energy. The RS theorem tells us that x is cyclic for $\mathcal{A}(O')$, and therefore, separating for $\mathcal{A}(O')'$. But by microcausality, $\mathcal{A}(O) \subseteq \mathcal{A}(O')'$, hence x must be separating for the subalgebra $\mathcal{A}(O)$ as well. Thus it is an immediate corollary to the RS theorem that

If x has bounded energy, then x is separating for any local algebra $\mathcal{A}(O)$ with $O' \neq \emptyset$.

It is this corollary that prompted Streater and Wightman's remark. But what has it got to do with thinking of the field system $\mathcal{A}(O)$ as isolated? For a start, we can now show that the local restriction $\rho_x|_{\mathcal{A}(O)}$ of a state with bounded energy is always a highly 'noisy' mixed state. Recall that a state ω on a von Neumann algebra $\mathcal{R}$ is a component of another state ρ if there is a third state τ such that $\rho = \lambda\omega + (1 - \lambda)\tau$ with $\lambda \in (0, 1)$ (Van Fraassen 1991, p. 161). We are going to show that $\rho_x|_{\mathcal{A}(O)}$ has a *norm* dense set of components in the state space of $\mathcal{A}(O)$.

Once again, the point is quite general. Let $\mathcal{R}$ be any von Neumann algebra, x be separating for $\mathcal{R}$, and let ω be an arbitrary state of $\mathcal{R}$. We must find a sequence $\{\omega_n\}$ of states of $\mathcal{R}$ such that each ω_n is a component of $\rho_x|_{\mathcal{R}}$ and $\|\omega_n - \omega\| \to 0$. Since $\mathcal{R}$ has a separating vector, it follows that every state of $\mathcal{R}$ is a vector state (KR 1997, Thm 7.2.3). (That this should be so is not as surprising as it sounds. Again, if $\mathsf{H} = \mathsf{H}_A \otimes \mathsf{H}_B$, and $\dim \mathsf{H}_A \geq \dim \mathsf{H}_B$, then as we have seen, the B subalgebra possesses a separating vector. But it is also easy to see, in this case, that every state on $I \otimes \mathcal{B}(\mathsf{H}_B)$ is the reduced density operator obtained from a pure state on $\mathcal{B}(\mathsf{H})$ determined by a vector in H.) In particular, there is a nonzero vector $y \in \mathsf{H}$

such that $\omega = \omega_y$. Since x is separating for $\mathcal{R}$, x is cyclic for $\mathcal{R}'$, therefore we may choose a sequence of operators $\{A_n\} \subseteq \mathcal{R}'$ so that $A_n x \to y$. Since $|A_n x - y| \to 0$, $\|\omega_{A_n x} - \omega_y\| \to 0$. We claim now that each $\omega_{A_n x}$ is a component of $\rho_x|_{\mathcal{R}}$. Indeed, for any positive element $B^*B \in \mathcal{R}$, we have:

$$\begin{aligned} \langle A_n x, B^*BA_n x\rangle &= \langle x, A_n^*A_nB^*Bx\rangle = \langle Bx, A_n^*A_nBx\rangle && (14)\\ &\leq |A_n^*A_n|\langle Bx, Bx\rangle = |A_n|^2\langle x, B^*Bx\rangle. && (15) \end{aligned}$$

Thus,

$$\omega_{A_n x}(B^*B) = \frac{\langle A_n x, B^*BA_n x\rangle}{|A_n x|^2} \leq \frac{|A_n|^2}{|A_n x|^2}\,\rho_x(B^*B). \tag{16}$$

If we now take $\lambda = |A_n x|^2/|A_n|^2 \in (0,1)$, and consider the linear functional τ on $\mathcal{R}$ given by $\tau = (1-\lambda)^{-1}(\rho_x|_{\mathcal{R}} - \lambda\omega_{A_n x})$, then (16) implies that τ is a state (in particular, positive), and we see that $\rho_x|_{\mathcal{R}} = \lambda\omega_{A_n x} + (1-\lambda)\tau$ as required. (This result also holds more generally for states ρ of $\mathcal{R}$ that are faithful, i.e., $\rho(Z) = 0$ entails $Z = 0$ for any positive $Z \in \mathcal{R}$; see the first part of the proof of Summers and Werner, 1988, Thm. 2.1.)

So bounded energy states are, locally, highly mixed. And such states are far from special — they lie norm dense in the pure state space of $\mathcal{B}(\mathsf{H})$. To see this, just recall that it is part of the spectral theorem for the global Hamiltonian that $E([0,n])$ converges strongly to the identity as $n \to \infty$. Thus we may approximate any vector $y \in \mathsf{H}$ by the sequence of bounded energy states $\{E([0,n])y/|E([0,n])y|\}_{n=0}^{\infty}$. Since there are so many bounded energy states of the field, that are locally so 'noisy', Streater and Wightman's comment is entirely warranted. But somewhat more can be said. As we saw with our toy example in Section 1, when a local subsystem of a global system in a pure state is itself in a mixed state, this is a sign of that subsystem's entanglement with its environment. And there is entanglement lurking in bounded energy states too. But, first, we need to take a closer look at the operational implications of local cyclicity.

If a vector x is cyclic for $\mathcal{R}$, then for any $y \in \mathcal{H}$, there is a sequence $A_n \in \mathcal{R}$ such that $A_n x \to y$. Thus for any $\epsilon > 0$ there is an $A \in \mathcal{R}$ such that $\|\rho_{Ax} - \rho_y\| < \epsilon$. However, ρ_{Ax} is just the state one gets by applying the pure operation given by the Kraus operator $K = A/|A| \in \mathcal{R}$ to ρ_x. It follows that if x is cyclic for $\mathcal{R}$, one can get arbitrarily close in norm to any other pure state of $\mathcal{B}(H)$ by applying an appropriate pure local operation in $\mathcal{R}$ to ρ_x. In particular, pure operations on the vacuum Ω within a local region O, no matter how small, can prepare essentially any global state of the field. As Haag emphasizes, to do this the operation must "judiciously exploit the small but nonvanishing long distance correlations which exist in the vacuum" (1992, p. 102). This, as Redhead (1995) has argued by analogy to the singlet state,

is made possible by the fact that the vacuum is highly entangled (cf. Clifton *et al* 1998). But the first puzzle we need to sort out is that it looks as though entirely *physical* operations in O can change the global state, in particular the vacuum Ω, to any desired state! (For example, Segal and Goodman (1965) have called this "bizarre" and "physically quite surprising", sentiments echoed recently by Fleming who calls it "amazing!" (1999).)

Redhead's analysis of the cyclicity of the singlet state $1/\sqrt{2}(a^+ \otimes b^- - a^- \otimes b^+)$ for the subalgebra $\mathcal{B}(\mathbb{C}^2_A) \otimes I$ is designed to remove this puzzle (*ibid*, p. 128). (Note that in this simple 2×2-dimensional case, he could equally well have chosen *any* entangled state, since they are all separating for $I \otimes \mathcal{B}(\mathbb{C}^2_B)$.) Redhead writes:

> "...we want to distinguish clearly two senses of the term "operation". Firstly there are physical operations such as making measurements, selecting subensembles according to the outcome of measurements, and mixing ensembles with probabilistic weights, and secondly there are the mathematical operations of producing superpositions of states by taking linear combinations of pure states produced by appropriate selective measurement procedures. These superpositions are of course quite different from the mixed states whose preparation we have listed as a physical operation" (1995, pp. 128-9).

Note that, in stark contrast to our discussion in the previous section, Redhead counts selecting subensembles and mixing as physical operations; it is only the operation of superposition that warrants the adjective 'mathematical'. When he explains why it is possible that x can be cyclic, Redhead first notes (*ibid*, p. 129) that the four basis states

$$a^+ \otimes b^-,\ a^- \otimes b^-,\ a^- \otimes b^+,\ a^+ \otimes b^-, \tag{17}$$

are easily obtained by the physical operations of applying projections and unitary transformations to the singlet state, and exploiting the fact that the singlet strictly correlates σ_a with σ_b. He goes on:

> "But *any* state for the joint system is some linear combination of these four states, so by the *mathematical* operation of linear combination, we can see how to generate an arbitrary state in $\mathsf{H}_1 \otimes \mathsf{H}_2$ from physical operations performed on particle one. But all the operations we have described can be represented in the algebra of operators on H_1 (extended to $\mathsf{H}_1 \otimes \mathsf{H}_2$)" (*ibid*, p. 129).

Now, while Redhead's explanation of why it is mathematically possible for x to be cyclic is perfectly correct, he actually misses the mark when it comes to the physical interpretation of cyclicity. The point is that superposition *of states* is a red-herring.

Certainly a superposition of the states in (17) could not be prepared by physical operations confined to the A system. But, as Redhead himself notes in the final sentence above, one can get the same *effect* as superposing those states by acting on x with an operator of form $A \otimes I$ in the subalgebra $\mathcal{B}(\mathbb{C}_A^2) \otimes I$ — an operator that is itself a 'superposition' of other operators in that algebra. What Redhead fails to point out is that the action of this operator on x *does have a local physical interpretation*: as we have seen, it is a Kraus operator that represents the outcome of a generalized positive operator valued measurement on the A system. The key to the puzzle is, rather, that this positive operator valued measurement will generally have to be *selective*. For one certainly could never, with nonselective operations on A alone, get as close as one likes to any state vector in $\mathbb{C}_A^2 \otimes \mathbb{C}_B^2$ (otherwise all state vectors would induce the same state on $I \otimes \mathcal{B}(\mathbb{C}_B^2)$!). We conclude that the correct way to view the physical content of cyclicity is that changes in the global state are partly due to an experimenter's ability to perform a generalized measurement on A, and partly due (*pace* Redhead) to the purely conceptual operation of selecting a subensemble based on the outcome of the experimenter's measurement together with the consequent 'change' in the state of B via the EPR correlations between A and B.

One encounters the same interpretational pitfall concerning the cyclicity of the vacuum in relation to localized states in AQFT. A global state of the field is said to be localized in O if its expectations on the algebra $\mathcal{A}(O')$ agree with vacuum expectation values (Haag, 1992, p. 102). Thus localized states are 'excitations' of the vacuum confined to O. In particular, $U\Omega$ will be a localized state whenever U is a unitary operator taken from $\mathcal{A}(O)$ (since unitary operations are nonselective). But every element of a C^*-algebra is a finite linear combination of unitary operators (KR 1997, Thm. 4.1.7). Since Ω is cyclic for $\mathcal{A}(O)$, this means we must be able to approximate any global state by linear superpositions of vectors describing states localized in O — even approximate states that are localized in regions spacelike separated with O! Haag, rightly cautious, calls this a "(superficial) paradox" (1992, p. 254), but he fails to put his finger on its resolution: while unitary operations are nonselective, a local operation in $\mathcal{A}(O)$ given by a Kraus operator that is a linear combination of local unitary operators will generally be *selective*. (Haag *does* make the interesting point out that only a proper subset of the state space of a field can be approximated if we restrict ourselves to local operations that involve a physically reasonable expenditure of energy. But we do not share the view of Schroer (1999) that this point by itself reconciles the RS theorem with 'common sense'.)

The (common) point of the previous two paragraphs is perhaps best summarized as follows. Both Redhead and Haag would agree that unitary Kraus operators in $\mathcal{A}(O)$ give rise to purely physical operations in the local region O. But there are many Kraus operators in $\mathcal{A}(O)$ that do not represent purely physical operations in O insofar as they are selective. Since every Kraus operator is a linear superposition

of unitary operators, it follows that "superposition of local operations" does not preserve (pure) physicality. Redhead is right that the key to diffusing the paradox is in noting that superpositions are involved — but it is essential to understand these superpositions as occurring locally in $\mathcal{A}(O)$, not in the Hilbert space.

Our next order of business is to supply the rigorous argument behind Redhead's intuition about the connection between cyclicity and entanglement. The point, again, is quite general: for any two commuting nonabelian von Neumann algebras $\mathcal{R}_A$ and $\mathcal{R}_B$, and any state vector x cyclic for $\mathcal{R}_A$ (or $\mathcal{R}_B$), ρ_x will be entangled across the algebras (Halvorson and Clifton, 2000, Prop. 2). For suppose, in order to extract a contradiction, that ρ_x is *not* entangled. Then as we have seen, operations on ρ_x that are local to $\mathcal{R}_A$ cannot turn that state into an entangled state across $(\mathcal{R}_A, \mathcal{R}_B)$. Yet, by the cyclicity of x, we know that we can apply pure operations to ρ_x, that are local to $\mathcal{R}_A$ (or $\mathcal{R}_B$), and approximate in norm (and hence weak-* approximate) any other vector state of $\mathcal{R}_{AB}$. It follows that no vector state of $\mathcal{R}_{AB}$ could be entangled across $(\mathcal{R}_A, \mathcal{R}_B)$, and the same goes for all its mixed states, which lie in the the norm closed convex hull of the vector states. But this means that $\mathcal{R}_{AB}$ would possess *no* entangled states at all — in flat contradiction with the fact that neither $\mathcal{R}_A$ nor $\mathcal{R}_B$ is abelian.

Returning to the context of AQFT, if we now consider *any* two spacelike separated field systems, $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$, then the argument we just gave establishes that the dense set of field states bounded in the energy will *all* be entangled across the regions (O_A, O_B). (Note that the fact that $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ are nonabelian is *itself* a consequence of the RS theorem. For if, say, $\mathcal{A}(O_A)$ were abelian, then since by the RS theorem that algebra possesses a cyclic vector, it must be a maximal abelian subalgebra of $\mathcal{B}(\mathsf{H})$ (KR 1997, Cor. 7.2.16). The same conclusion would have to follow for any subregion $\tilde{O}_A \subset O_A$ whose closure is a proper subset of O_A. And this, by isotony, would lead to the absurd conclusion that $\mathcal{A}(\tilde{O}_A) = \mathcal{A}(O_A)$, which is easily shown to be inconsistent with the axioms of AQFT (Horuzy, 1988, Lemma 1.3.10).) However, by itself this result does not imply that Alice cannot destroy a bounded energy state x's entanglement across (O_A, O_B) by performing local operations in O_A. In fact, Borchers (1965, Cor. 7) has shown that any vector state of form Ax for any nontrivial $A \in \mathcal{A}(O_A)$ *never* has bounded energy (nor is 'analytic' in the energy). So it seems that all Alice needs to do is perform any pure operation within O_A and the resulting state, because it is no longer subject to the RS theorem, need no longer be entangled across (O_A, O_B).

However, the RS theorem gives only a sufficient, *not* a necessary, condition for a state x of the field to be cyclic for $\mathcal{A}(O_A)$. And notwithstanding that no pure operation Alice performs can preserve boundedness in the energy, *almost all* the pure operations she could perform *will* preserve the state's cyclicity! The reason is, once again, quite general. Again let $\mathcal{R}_A$ and $\mathcal{R}_B$ be two commuting nonabelian von

Neumann algebras, suppose x is cyclic for $\mathcal{R}_A$, and consider the state induced by the vector Ax where $A \in \mathcal{R}_A$. Now every element in a von Neumann algebra is the strong limit of invertible elements in the algebra (Dixmier and Maréchal, 1971, Prop. 1). Therefore, there is a sequence of invertible operators $\{\tilde{A}_n\} \subseteq \mathcal{R}_A$ such that $\tilde{A}_n x \to Ax$, i.e., $\|\rho_{\tilde{A}_n x} - \rho_{Ax}\| \to 0$. Notice, however, that since each $\tilde{A}_n$ is invertible, each vector $\tilde{A}_n x$ is again cyclic for $\mathcal{R}_A$, because we can 'cycle back' to x by applying to $\tilde{A}_n x$ the inverse operator $\tilde{A}_n^{-1} \in \mathcal{R}_A$, and from there we know, by hypothesis, that we can cycle with elements of $\mathcal{R}_A$ arbitrarily close to any other vector in H. It follows that, even though Alice may *think* she has applied the pure operation given by some Kraus operator $A/|A|$ to x, she could well have *actually* applied an invertible Kraus operation given by one of the operators $\tilde{A}_n/|\tilde{A}_n|$ in a strong neighborhood of $A/|A|$. And if she actually did this, then she certainly would *not* disentangle x, because she would not have succeeded in destroying the *cyclicity* of the field state for her local algebra. We could, of course, give Alice the freedom to employ more general mixing operations in O_A. But as we saw in the last section, it is far from clear whether a mixing operation should count as a successful disentanglement when all the states that are mixed by her operation are themselves entangled — or at least not *known* by Alice to be disentangled (given her practical inability to specify exactly which Kraus operations go into the pure operations of her mixing process).

Besides this, there is a more fundamental practical limitation facing Alice, even if we allow her any local operation she chooses. If, as we have seen, we can approximate the result of acting on x with any given operator in von Neumann algebra $\mathcal{R}$ by acting on x with an invertible operator that preserves x's cyclicity, then the set of all such 'invertible actions' on x must itself produce a dense set of vector states, given that $\{Ax : A \in \mathcal{R}\}$ is dense. It follows that if a von Neumann algebra possesses even just one cyclic vector, it must possess a dense set of them (Dixmier and Maréchal, 1971, Lemma 4; cf. Clifton *et al* 1998). Now consider, again, the general situation of two commuting nonabelian algebras $\mathcal{R}_A$ and $\mathcal{R}_B$, where either algebra possesses a cyclic vector, and hence a dense set of such. If, in addition, the algebra $\mathcal{R}_{AB}$ possesses a separating vector, then *all* states of that algebra will be vector states, a *norm* dense set of which must therefore be entangled across $(\mathcal{R}_A, \mathcal{R}_B)$. And since the entangled states of $\mathcal{R}_{AB}$ are open in the weak-* topology, they must be open in the (stronger) norm topology too — so we are dealing with a truly generic set of states. It follows, quite independently of the RS theorem, that

Generic Result: If $\mathcal{R}_A$ and $\mathcal{R}_B$ are commuting nonabelian von Neumann algebras either of which possesses a cyclic vector, and $\mathcal{R}_{AB}$ possesses a separating vector, then the generic state of $\mathcal{R}_{AB}$ will be entangled across $(\mathcal{R}_A, \mathcal{R}_B)$.

The role that the RS theorem plays is to guarantee that the antecedent conditions of this Generic Result are satisfied whenever we consider spacelike separated regions

(and corresponding algebras) satisfying $(O_A \cup O_B)' \neq \emptyset$. This is a very weak requirement, which is satisfied, for example, when we assume both regions are bounded in spacetime. In that case, in order to be *certain* that her local operation in O_A (pure or mixed) produced a disentangled state, Alice would need the extraordinary ability to distinguish the state of $\mathcal{A}_{AB}$ which results from her operation from the generic set states of $\mathcal{A}_{AB}$ that are entangled!

Finally, while we noted in our introduction the irony that limitations on disentanglement arise precisely when one considers *relativistic* quantum theory, the practical limitations we have just identified — as opposed to the *intrinsic* limits on disentanglement which are the subject of the next section — are not characteristic of AQFT alone. In particular, the existence of locally cyclic states does not depend on field theory. As we have seen, both the A and B subalgebras of $\mathcal{B}(\mathsf{H}_A \otimes \mathsf{H}_B)$ possess a cyclic vector just in case $\dim \mathsf{H}_A = \dim \mathsf{H}_B$. Indeed, operator algebraists so often find themselves dealing with von Neumann algebras that, together with their commutants, possess a cyclic vector, that such algebras are said by them to be in 'standard form'. So we should not think that local cyclicity is somehow peculiar to the states of local quantum fields.

Neither is it the case that our Generic Result above finds its only application in quantum *field* theory. For example, consider the infinite-by-infinite state space $\mathsf{H}_A \otimes \mathsf{H}_B$ of any two nonrelativistic particles, ignoring their spin degrees of freedom. Take the tensor product with a third auxiliary infinite-dimensional Hilbert space $\mathsf{H}_A \otimes \mathsf{H}_B \otimes \mathsf{H}_C$. Then obviously $\infty = \dim \mathsf{H}_C \geq \dim(\mathsf{H}_A \otimes \mathsf{H}_B) = \infty$, whence the C subalgebra possesses a cyclic vector, which is therefore separating for the $A + B$ algebra. On the same dimensional grounds, both the A and B subalgebras possess cyclic vectors of their own. So our Generic Result applies immediately yielding the conclusion that a typical state of $A + B$ will be entangled (cf. Clifton and Halvorson, 2000).

Nor should we think of local cyclicity or the applicability of our Generic Result as peculiar to standard *local* quantum field theory. After noting that the local cyclicity of the vacuum in AQFT was a "great, counterintuitive, surprise" (p. 4) when it was first proved, Fleming (1999) proposes, instead, to build up local algebras associated with bounded open spatial sets within hyperplanes from raising and lowering operators associated with nonlocal Newton-Wigner position eigenstates — a proposal that goes back at least as far as Segal (1964). Fleming then observes, as did Segal (1964, p. 143), that the resulting vacuum state will *not* be entangled nor cyclic for any such local algebra. Nevertheless, as Segal points out, each Segal-Fleming local algebra will be isomorphic to the algebra $\mathcal{B}(\mathsf{H})$ of all bounded operators on an *infinite*-dimensional Hilbert space H, and algebras associated with spacelike-separated regions in the same hyperplane commute. It follows that if we take any two spacelike separated bounded open regions O_A and O_B lying in the same hyperplane, $[\mathcal{A}(O_A) \cup \mathcal{A}(O_B)]''$ will be

naturally isomorphic to $\mathcal{B}(\mathsf{H}_A) \otimes \mathcal{B}(\mathsf{H}_B)$ (Horuzhy 1988, Lemma 1.3.28), and the result of the previous paragraph applies. So Fleming's 'victory' over the RS theorem of standard local quantum field theory rings hollow. Even though the Newton-Wigner vacuum is not itself entangled or locally cyclic across the regions (O_A, O_B), it will be indistinguishable from globally pure states of the Newton-Wigner field that are! (For further discussion of the Segal-Fleming approach to quantum fields, see Halvorson (2000).)

On the other hand, generic entanglement is certainly not to be expected in every quantum-theoretic context. For example, if we ignore external degrees of freedom, and just consider the spins of two particles with joint state space $\mathsf{H}_A \otimes \mathsf{H}_B$, where both spaces are nontrivial and *finite*-dimensional, then the Generic Result no longer applies. Taking the product with a third auxiliary Hilbert space H_C does not work, because in order for the $A + B$ subalgebra to have a separating vector we would need $\dim H_C \geq \dim H_A \dim H_B$, but for either the A or B subalgebras to possess a cyclic vector we would *also* need that either $\dim H_A \geq \dim H_B \dim H_C$ or $\dim H_B \geq \dim H_A \dim H_C$ — both of which contradict the fact H_A and H_B are nontrivial and finite-dimensional. (In fact, it can be shown that the spins of any pair of particles are *not* generically entangled, unless of course we ignore their mixed spin states; see Clifton and Halvorson, 2000 for further discussion.) The point is that while the conditions for generic entanglement may or may not obtain in *any* quantum-theoretical context — depending on the observables and dimensions of the state spaces involved — the beauty of the RS theorem is that it allows us to deduce that generic entanglement between bounded open spacetime regions *must* obtain just by making some very general and natural assumptions about what should count as a physically reasonable relativistic quantum field theory.

4. Type III von Neumann Algebras and Intrinsic Entanglement

Though it is not known to follow from the general axioms of AQFT (cf. Kadison, 1963), all known concrete models of the axioms are such that the local algebras associated with bounded open regions in M are type III factors (Horuzy, 1988, pgs. 29, 35; Haag, 1992, Sec. V.6). We start by reviewing what precisely is meant by the designation 'type III factor'.

A von Neumann algebra $\mathcal{R}$ is a factor just in case its center $\mathcal{R} \cap \mathcal{R}'$ consists only of multiples of the identity. It is easy to verify that this is equivalent to $(\mathcal{R} \cup \mathcal{R}')'' = \mathcal{B}(\mathsf{H})$, thus $\mathcal{R}$ induces a 'factorization' of the total Hilbert space algebra $\mathcal{B}(\mathsf{H})$ into two subalgebras which together generate that algebra.

To understand what 'type III' means, a few further definitions need to be absorbed. A partial isometry V is an operator on a Hilbert space H that maps some particular closed subspace $C \subseteq \mathsf{H}$ isometrically onto another closed subspace $C' \subseteq \mathsf{H}$, and maps $C^{\perp}$ to zero. (Think of V as a 'hybrid' unitary/projection operator.) Given the set of projections in a von Neumann algebra $\mathcal{R}$, we can define the following equivalence relation on this set: $P \sim Q$ just in case there is a partial isometry $V \in \mathcal{R}$ that maps the range of P onto the range of Q. (It is important to notice that this definition of equivalence is relative to the particular von Neumann algebra $\mathcal{R}$ that the projections are considered to be members of.) For example, any two infinite-dimensional projections in $\mathcal{B}(\mathsf{H})$ are equivalent (when H is separable), including projections one of whose range is properly contained in the other (cf. KR 1997, Cor. 6.3.5). A nonzero projection $P \in \mathcal{R}$ is called abelian if the von Neumann algebra $P\mathcal{R}P$ acting on the subspace $P\mathsf{H}$ (with identity P) is abelian. One can show that the abelian projections in a factor $\mathcal{R}$ are exactly the atoms in its projection lattice (KR 1997, Prop. 6.4.2). For example, the atoms of the projection lattice of $\mathcal{B}(\mathsf{H})$ are all its one-dimensional projections, and they are all (trivially) abelian, whereas it is clear that higher-dimensional projections are not. Finally, a projection $P \in \mathcal{R}$ is called infinite (relative to $\mathcal{R}$!) when it is equivalent to another projection $Q \in \mathcal{R}$ such that $Q < P$, i.e., Q projects onto a proper subspace of the range of P. One can also show that any abelian projection in a von Neumann algebra must be *finite*, i.e., not infinite (KR 1997, Prop. 6.4.2).

A type I von Neumann factor is now defined as one that possesses an abelian projection. For example, $\mathcal{B}(\mathsf{H})$ for any Hilbert space H is always type I, and, indeed, every type I factor arises as the algebra of all bounded operators on some Hilbert space (KR 1997, Thm 6.6.1). On the other hand, a factor is type III if all its nonzero projections are infinite and equivalent. In particular, this entails that the algebra itself is not abelian, nor could it even possess an abelian projection — which would have to be finite. And since a type III factor contains no abelian projections, its projection lattice cannot have any atoms. Another fact about type III algebras is that they *always* possess a vector that is both cyclic and separating (Sakai, 1971, Cor. 2.9.28). Therefore we know that type III algebras will always possess a dense set of cyclic vectors, and that all their states will be vector states. *Notwithstanding this*, type III algebras possess *no* pure states, as a consequence of the fact that they lack atoms.

To get some feeling for why this is the case — and for the general connection between the failure of the projection lattice of an algebra to possess atoms and its failure to possess pure states — let $\mathcal{R}$ be any non-atomic von Neumann algebra possessing a separating vector (so all of its states are vector states), and let ρ_x be any state of $\mathcal{R}$. We shall need two further definitions. The support projection, S_x, of ρ_x in $\mathcal{R}$ is defined to be the meet of all projections $P \in \mathcal{R}$ such that $\rho_x(P) = 1$.

(So S_x is the smallest projection in $\mathcal{R}$ that ρ_x 'makes true'.) The left-ideal, I_x, of ρ_x in $\mathcal{R}$ is defined to be the set of all $A \in \mathcal{R}$ such that $\rho_x(A^*A) = 0$. Now since S_x is not an atom, there is some nonzero $P \in \mathcal{R}$ such that $P < S_x$. Choose any vector y in the range of P (noting it follows that $S_y \leq P$). We shall first show that I_x is a proper subset of I_y. So let $A \in I_x$. Clearly this is equivalent to saying that $Ax = 0$, or that x lies in the range of $N(A)$, the projection onto the null-space of A. $N(A)$ itself lies $\mathcal{R}$ (KR 1997, Lemma 5.1.5 and Prop. 2.5.13), thus, $\rho_x(N(A)) = 1$, and accordingly $S_x \leq N(A)$. But since $S_y \leq P < S_x$, we also have $\rho_y(N(A)) = 1$. Thus, y too lies in the range of $N(A)$, i.e., $Ay = 0$, and therefore $A \in I_y$. To see that the inclusion $I_x \subset I_y$ is proper, note that since $(y, S_y y) = 1$, $(y, [I - S_y]^2 y) = 0$, and thus $I - S_y \in I_y$. However, certainly $I - S_y \notin I_x$, for the contrary would entail that $(x, S_y x) = 1$, in other words, $S_x \leq S_y \leq P < S_x$ — a contradiction. We can now see, finally, that ρ_x cannot be pure. For, quite generally, the pure states of a von Neumann algebra $\mathcal{R}$ determine *maximal* left-ideals in $\mathcal{R}$ (KR 1997, Thm. 10.2.10), yet we have just shown, under the assumption that $\mathcal{R}$ is non-atomic, that $I_x \subset I_y$.

The fact that every state of a type III algebra $\mathcal{R}$ is mixed throws an entirely new wrench into the works of the ignorance interpretation of mixtures. (To our knowledge, Van Aken (1985) is the only philosopher of quantum theory to have noticed this.) Not only is there no preferred way to pick out components of a mixture, but the components of states of $\mathcal{R}$ will always *themselves* be mixtures. Thus, it is impossible to understand the preparation of such a mixture in terms of mixing pure states — the states of $\mathcal{R}$ are always irreducibly or *intrinsically* mixed. Note, however, that while the states of type III factors fit this description, so do the states of certain *abelian* von Neumann algebras. For example, the 'multiplication' algebra $\mathcal{M} \subseteq \mathcal{B}(L_2(\mathbb{R}))$ of all bounded functions of the position operator for a single particle lacks atomic projections because position has no eigenvectors. Moreover, all the states of $\mathcal{M}$ are vector states, because any state vector that corresponds to a wavefunction whose support is the whole of $\mathbb{R}$ is separating for $\mathcal{M}$. Thus the previous paragraph's argument applies equally well to $\mathcal{M}$.

Of course no properly *quantum* system has an abelian algebra of observables, and, as we have already noted, systems with abelian algebras are never entangled with other systems. This makes the failure of a type III factor $\mathcal{R}$ to have pure states importantly different from that failure in the case of an abelian algebra. Because $\mathcal{R}$ is *non*abelian, and taking the commutant preserves type (KR 1997, Thm. 9.1.3) so that $\mathcal{R}'$ will also be nonabelian, one suspects that any pure state of $(\mathcal{R} \cup \mathcal{R}')'' = \mathcal{B}(\mathsf{H})$ — which must restrict to an intrinsically mixed state on both subalgebras $\mathcal{R}$ and $\mathcal{R}'$ — has to be *intrinsically entangled* across $(\mathcal{R}, \mathcal{R}')$. And that intuition is exactly right; indeed, one can show that there are not even any *product* states across $(\mathcal{R}, \mathcal{R}')$ (Summers 1990, p. 213). And, of course, if there are no unentangled states across $(\mathcal{R}, \mathcal{R}')$, then the infamous distinction, some have argued is important to preserve,

between so-called 'improper' mixtures that arise by restricting an entangled state to a subsystem, and 'proper' mixtures that do not, becomes *irrelevant*.

Even more interesting is the fact that in all known models of AQFT, the local algebras are 'type III_1'. It would take us too far afield to explain the standard sub-classification of factors presupposed by the subscript '1'. We wish only to draw attention to an equivalent characterization of type III_1 algebras established by Connes and Størmer (1978, Cor. 6): A factor $\mathcal{R}$ acting standardly on a (separable) Hilbert space is type III_1 just in case for *any two* states ρ, ω of $\mathcal{B}(\mathsf{H})$, and any $\epsilon > 0$, there are unitary operators $U \in \mathcal{R}$, $U' \in \mathcal{R}'$ such that $\|\rho - \omega^{UU'}\| < \epsilon$. Notice that this result immediately implies that there can be no unentangled states across $(\mathcal{R}, \mathcal{R}')$; for, if some ω were not entangled, it would be impossible to act on this state with local unitary operations in $\mathcal{R}$ and $\mathcal{R}'$ and get arbitrarily close to the states that *are* entangled across $(\mathcal{R}, \mathcal{R}')$. Furthermore — and this is the interesting fact — the Connes-Størmer characterization immediately implies the impossibility of distinguishing in any reasonable way between the different degrees of entanglement that states might have across $(\mathcal{R}, \mathcal{R}')$. For it is a standard assumption in quantum information theory that all reasonable measures of entanglement must be *invariant* under unitary operations on the separate entangled systems (cf. Vedral *et al*, 1997), and presumably such a measure should assign close degrees of entanglement to states that are close to each other in norm. In light of the Connes-Størmer characterization, imposition of both these requirements forces triviality on any proposed measure of entanglement across $(\mathcal{R}, \mathcal{R}')$. Of course, the standard von Neumann entropy measure we discussed in Section 1. is norm continuous, and, because of the unitary invariance of the trace, this measure is invariant under unitary operations on the component systems. But in the case of a type III factor $\mathcal{R}$, that measure, as we should expect, is *not* available. Indeed, the state of a system described by $\mathcal{R}$ cannot be represented by any density operator *in* $\mathcal{R}$ because $\mathcal{R}$ cannot contain compact operators, like density operators, whose spectral projections are all finite!

The above considerations have particularly strong physical implications when we consider local algebras associated with diamond regions in M, i.e., regions given by the intersection of the timelike future of a given spacetime point p with the timelike past of another point in p's future. When $\diamondsuit \subseteq M$ is a diamond, it can be shown in many models of AQFT, including for *non*interacting fields, that $\mathcal{A}(\diamondsuit') = \mathcal{A}(\diamondsuit)'$ (Haag 1992, Sec. III.4.2). Thus every global state of the field will be intrinsically entangled across $(\mathcal{A}(\diamondsuit), \mathcal{A}(\diamondsuit'))$, and it is never possible to think of the field system in a diamond region $\diamondsuit$ as disentangled from that of its spacelike complement. Though he does not use the language of entanglement, this is precisely the reason for Haag's remark that field systems are always open. In particular, Alice would have *no hope whatsoever* of using local operations in $\diamondsuit$ to disentangle that region's state from that of the rest of the world.

Suppose, however, that Alice has only the more limited goal of disentangling a state of the field across some isolated pair of *strictly* spacelike-separated regions (O_A, O_B), i.e., regions which remain spacelike separated when either is displaced by an arbitrarily small amount. It is also known that in many models of AQFT the local algebras possess the split property: for any bounded open $O \subseteq M$, and any larger region $\tilde{O}$ whose interior contains the closure of O, there is a type I factor $\mathcal{N}$ such that $\mathcal{A}(O) \subset \mathcal{N} \subset \mathcal{A}(\tilde{O})$ (Bucholz 1974, Werner 1987). This implies that the von Neumann algebra generated by a pair of algebras for strictly spacelike separated regions is isomorphic to their tensor product and, as a consequence, that there *are* product states across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$ (cf. Summers 1990, pgs. 239-40). Since, therefore, not every state of $\mathcal{A}_{AB}$ is entangled, we might hope that whatever the global field state is, Alice could *at least in principle* perform an operation in O_A on that state that disentangles it across (O_A, O_B). However, we are now going use the fact that $\mathcal{A}(O_A)$ lacks abelian projections to show that a norm dense set of entangled states of $\mathcal{A}_{AB}$ cannot be disentangled by any pure local operation performed in $\mathcal{A}(O_A)$.

Let ρ_x be any one of the norm dense set of entangled states of $\mathcal{A}_{AB}$ induced by a vector $x \in \mathsf{H}$ cyclic for $\mathcal{A}(O_B)$, and let $K \in \mathcal{A}(O_A)$ be an arbitrary Kraus operator. (Observe that $\rho_x^K \neq 0$ because x is separating for $\mathcal{A}(O_B)'$ — which includes $\mathcal{A}(O_A)$ — and $K^*K \in \mathcal{A}(O_A)$ is positive.) Suppose, for the purposes of extracting a contradiction, that ω_x^K is not also entangled. Let Ky, with $y \in \mathsf{H}$, be any nonzero vector in the range of K. Then, since x is cyclic for $\mathcal{A}(O_B)$, we have, for some sequence $\{B_i\} \subseteq \mathcal{A}(O_B)$, $Ky = K(\lim B_i x) = \lim(B_i K x)$, which entails $\|(\omega_x^K)^{B_i/|B_i|} - \omega_{Ky}\| \to 0$. Since ω_x^K is not entangled across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$, and the local pure operations on $\mathcal{A}(O_B)$ given by the Kraus operators $B_i/|B_i|$ cannot create entanglement, we see that ω_{Ky} is the norm (hence weak-*) limit of a sequence of unentangled states and, as such, is not itself entangled either. Since y was arbitrary, it follows that every nonzero vector in the range of K induces an unentangled state across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$. Obviously, the same conclusion follows for any nonzero vector in the range of $R(K)$ — the range projection of K — since the range of the latter lies dense in that of the former.

Next, consider the von Neumann algebra

$$\mathcal{C}_{AB} \equiv [R(K)\mathcal{A}(O_A)R(K) \cup R(K)\mathcal{A}(O_B)R(K)]'' \tag{18}$$

acting on the Hilbert space $R(K)\mathsf{H}$. Since $K \in \mathcal{A}(O_A)$, $R(K) \in \mathcal{A}(O_A)$ (KR 1997, p. 309), and thus the subalgebra $R(K)\mathcal{A}(O_A)R(K)$ cannot be abelian — on pain of contradicting the fact that $\mathcal{A}(O_A)$ has no abelian projections. And neither is $R(K)\mathcal{A}(O_B)R(K)$ abelian. For since $\mathcal{A}(O_B)$ itself is nonabelian, there are $Y_1, Y_2 \in \mathcal{A}(O_B)$ such that $[Y_1, Y_2] \neq 0$. And because our regions (O_A, O_B) are strictly spacelike separated, they have the Schlieder property: $0 \neq A \in \mathcal{A}(O_A), 0 \neq B \in \mathcal{A}(O_B)$ implies $AB \neq 0$ (Summers 1990, Thm. 6.7). Therefore,

$$[R(K)Y_1R(K), R(K)Y_2R(K)] = [Y_1, Y_2]R(K) \neq 0. \tag{19}$$

So we see that neither algebra occurring in $\mathcal{C}_{AB}$ is abelian; yet they commute, and so there must be at least one entangled state across those algebras. But this conflicts with the conclusion of the preceding paragraph! For the vector states of $\mathcal{C}_{AB}$ are precisely those induced by the vectors in the range of $R(K)$, and we deduced that these all induce unentangled states across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$. Therefore, by restriction, they all induce unentangled states across the algebra $\mathcal{C}_{AB}$. But if none of $\mathcal{C}_{AB}$'s vector states are entangled, it can possess *no* entangled states at all.

The above argument still goes through under the weaker assumption that Alice applies any mixed *projective* operation, i.e., any operation T corresponding to a standard von Neumann measurement associated with a mutually orthogonal set $\{P_i\} \in \mathcal{A}(O_A)$ of projection operators. For if we suppose, again for reductio, that $\rho_x^T = \sum_i \lambda_i \rho_x^{P_i}$ is not entangled across the regions, then since entanglement cannot be created by a further application to ρ_x^T of the local projective operation given by (say) $T_1(\cdot) = P_1(\cdot)P_1$, it follows that $(\rho_x^T)^{T_1} = (\rho_x^{T_1 \circ T}) = \rho_x^{P_1}$ must again be unentangled, and the above reasoning to a contradiction goes through *mutatis mutandis* with $K = P_1$. This is to be contrasted to the nonrelativistic case we considered in Section 1, where Alice *was* able to disentangle an arbitrary state of $\mathcal{B}(H_A \otimes H_B)$ by a nonselective projective operation on A. And a moment's reflection will reveal that that was possible precisely because of the availability of abelian projections in the algebra of her subsystem A. We have not, of course, shown that the above argument covers *arbitrary* mixing operations Alice might perform in O_A; in particular, positive-operator valued mixings, where the Kraus operators $\{K_i\}$ of a local operation T in O_A do not have mutually orthogonal ranges. However, although it would be interesting to know how far the result could be pushed, we have already expressed our reservations about whether arbitrary mixing operations should count as disentangling when none of the pure operations of which they are composed could possibly produce disentanglement on their own.

In summary:

There are many regions of spacetime within which no local operations can be performed that will disentangle that region's state from that of its spacelike complement, and within which no pure or projective operation on any one of a norm dense set of states can yield disentanglement from the state of any other strictly spacelike-separated region.

Clearly the advantage of the formalism of AQFT is that it allows us to see clearly just how much more deeply entrenched entanglement is in *relativistic* quantum theory. At the very least, this should serve as a strong note of caution to those who would quickly assert that quantum nonlocality cannot peacefully exist with relativity.

As far as what becomes of Einsteinian worries about the possibility of doing science in such a deeply entangled world, the split property of local algebras comes to the

rescue. For let us suppose Alice knows nothing more than that she wants to prepare some state ρ on $\mathcal{A}(O_A)$ for subsequent testing. (The following argument is simply an amplification of the reasoning in Werner 1987 and Summers 1990, Thm. 3.13.) Since there is a type I factor $\mathcal{N}$ satisfying $\mathcal{A}(O_A) \subset \mathcal{N} \subset \mathcal{A}(\tilde{O}_A)$ for any super-region $\tilde{O}_A$, and ρ is a vector state (when we assume $(O_A)' \neq \emptyset$), its vector representative defines a state on $\mathcal{N}$ that extends ρ and is, therefore, represented by some density operator D_ρ in the type I algebra $\mathcal{N}$. Now D_ρ is an infinite convex combination $\sum_i \lambda_i P_i$ of mutually orthogonal atomic projections in $\mathcal{N}$ satisfying $\sum_i P_i = I$ with $\sum_i \lambda_i = 1$. But each such projection is equivalent, *in the type* III *algebra* $\mathcal{A}(\tilde{O}_A)$, to the identity operator. Thus, for each i, there is a partial isometry $V_i \in \mathcal{A}(\tilde{O}_A)$ satisfying $V_i V_i^* = P_i$ and $V_i^* V_i = I$. Next, consider the nonselective operation T on $\mathcal{A}(\tilde{O}_A)$ given by Kraus operators $K_i = \sqrt{\lambda_i} V_i$, and fix an arbitrary $X \in \mathcal{A}(O_A)$. We claim that $T(X) = \rho(X)I$. Indeed, because each P_i is abelian in $\mathcal{N} \supseteq \mathcal{A}(O_A)$, the operator $P_i X P_i$ acting on $P_i \mathsf{H}$ can only be some multiple, c_i, of the identity operator P_i on $P_i \mathsf{H}$, and taking the trace of both sides of the equation

$$P_i X P_i = c_i P_i \tag{20}$$

immediately reveals that $c_i = \mathrm{Tr}(P_i X)$. Moreover, acting on the left of (20) with V_i^* and on the right with V_i, we obtain $V_i^* X V_i = \mathrm{Tr}(P_i X) I$, which yields the desired conclusion when multiplied by λ_i and summed over i. Finally, since $T(X) = \rho(X)I$ for all $X \in \mathcal{A}(O_A)$, obviously $\omega^T = \rho$ for all initial states ω of $\mathcal{A}(O_A)$. Thus, once we allow Alice to perform an operation like T that is *approximately* local to $\mathcal{A}(O_A)$ (choosing $\tilde{O}_A$ to approximate O_A as close as we like), she has the freedom to prepare any state of $\mathcal{A}(O_A)$ that she pleases.

Notice that, ironically, testing the theory is actually *easier* here than in nonrelativistic quantum theory! For we were able to exploit above the type III character of $\mathcal{A}(\tilde{O}_A)$ to show that Alice can always prepare her desired state on $\mathcal{A}(O_A)$ *nonselectively*, i.e., without ever having to sacrifice any members of her ensemble! Also observe that the result of her preparing operation T, because it is local to $\mathcal{A}(\tilde{O}_A)$, will always produce a product state across (O_A, O_B) when $O_B \subseteq (\tilde{O}_A)'$. That is, for any initial state ω across the regions, and all $X \in \mathcal{A}(O_A)$ and $Y \in \mathcal{A}(O_B)$, we have

$$\omega^T(XY) = \omega(T(X)Y) = \omega(\rho(X)Y) = \rho(X)\omega(Y). \tag{21}$$

So as soon as we allow Alice to perform *approximately* local operations on her field system, she *can* isolate it from entanglement with other strictly spacelike-separated field systems, while simultaneously preparing its state as she likes and with relative ease. God is subtle, but not malicious.

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